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Shape invariance for the bent brane with two scalar fields

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Abstract. In this paper, we study the braneworld scenarios in the presence of two real scalar fields coupled by gravity. The first-order formalism for the bent brane (for both de Sitter and anti-de Sitter geometry), leads us to discuss the shape invariance method in the bent brane systems. So, by using the fluctuations of metric and fields we obtain the Schrödinger equation. Then we factorize the corresponding Hamiltonian in terms of multiplication of the first-order differential operators. These first-order operators lead us to obtain the energy spectrum with the help of shape invariance method.

1 Introduction

In this paper we focus attention on the braneworld scenario described by five-dimensional space-time with warped geometry [1]. In that case, we consider branes in the five-dimensional de Sitter (dS) scenario with embedding four-dimensional AdS, Minkowski, or de Sitter (dS) geometry [2–6].

The first-order formalism in the braneworld scenario driven by one scalar field with embedded geometry of the AdS, M, or dS type is discussed in [1, 6]. By using [1] we can continue this process for the two fields coupled by gravity.

This model leads us to introduce the two functions $W(\phi,\chi)$ and $Z(\phi,\chi)$. By considering equations of motion in this model which allows for the cosmological constant, we can achieve two constraints for W and Z. Then, we relate the functions $W(\phi,\chi)$ and $Z(\phi,\chi)$ to the warp factor (A). We obtain the potential $V(\phi,\chi)$ in terms of the functions W and Z which lead us to two constraints.

With the help of this superpotential and fluctuations of the metric and fields one can obtain the corresponding Schrödinger equation. Also we note that supersymmetry (symmetry between the fermionic and bosonic degrees of freedom) has an important role [7–9] in analyzing the quantum mechanical systems, since it can study remarkable properties including the degeneracy structure of the energy spectrum and the relations among the energy spectra of the various Hamiltonians etc. In particular the energy eigenvalues are necessarily non-negative and the energy of the non-zero (zero) ground state is related to the broken (unbroken) supersymmetry. As we know, Infeld and Hull [10] studied the factorization method. They factorized the second-order equations to first-order equations. Gendenstein et al. considered the subject in the framework of the shape invariance symmetry as an important aspect of the quantum mechanical models. So, we factorize this Schrödinger equation in terms of a first-order equation, and finally this leads us to consider the formalism of shape invariance. On the other hand shape invariance [11–14] provides perhaps the most illuminating approach to exact solvability in quantum mechanics. Shape invariance arises when a quantum mechanical model is invariant under both a supersymmetry algebra with a central charge and an additional symmetry operator, analogous to the Laplace–Lenz vector [15–20].

This paper is structured as follows. Section 2 is devoted to a formulation of the bent brane model with two real scalar fields, whereas in Sect. 3 we review however the supersymmetry algebra with the central charge and shape invariance method. In Sect. 4 we take advantage of this method for obtaining energy spectrum and eigenstates in this geometry. The final section contains concluding remarks.

2 Formalism

We consider the action of two interacting fields in five dimensions; the model that we investigate is described by the action

$$S=\int \mathrm{d}^4x\,\mathrm{d}y\sqrt{|g|}\times\left(-\frac{1}{4}R+\frac{1}{2}\partial_a\phi\partial^a\phi+\frac{1}{2}\partial_a\chi\partial^a\chi-V(\phi,\chi)\right),\quad(1)$$

where ϕ and χ stand for real scalar fields, and we take $4\pi G=1$.

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The line element of the five-dimensional space-time can be written

$$\mathrm{d}s_5^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = \mathrm{e}^{2A}\,\mathrm{d}s_4^2 - \mathrm{d}y^2, \tag{2}$$

where $a, b = 0, 1, 2, 3, 4$, and e^{2A} is the warp factor.

dS_4^2 represent the four-dimensional metric:

$$\mathrm{d}s_4^2 = \mathrm{d}t^2 - \mathrm{e}^{2\sqrt{\Lambda}t}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right), \tag{3}$$

where Λ is a four-dimensional cosmological constant. We note that constant Λ is positive for de Sitter (dS_4) space-time, negative for anti-de Sitter (AdS_4) space-time and zero for Minkowski (M_4) space-time.

At first we assume that $\Lambda = 0$ and the functions A, ϕ, χ are $A(y)$,$\phi(y)$, $\chi(y)$.

From the Einstein and Euler–Lagrange equations we obtain,

$$\begin{aligned}
A'' &= -\frac{2}{3}(\phi'^2 + \chi'^2)\,,\\
A'^2 &= \frac{1}{6}(\phi'^2 + \chi'^2) - \frac{1}{3}V(\phi, \chi)\,,\\
V_\phi &= \phi'' + 4A'\phi'\,,\\
V_\chi &= \chi'' + 4A'\chi'\,,
\end{aligned} \tag{4}$$

where a prime denotes a derivative with respect to y, and

$$V_\phi = \frac{\mathrm{d}V}{\mathrm{d}\phi}\,, \quad V_\chi = \frac{\mathrm{d}V}{\mathrm{d}\chi}\,.$$

In order to obtain the first-order equation, we use [1]

$$\begin{aligned}
A' &= -\frac{1}{3}W\,,\\
\phi' &= \frac{1}{2}W_\phi\,,\\
\chi' &= \frac{1}{2}W_\chi\,.
\end{aligned} \tag{5}$$

From (4) and (5) the explicit form of the potential is

$$V(\phi, \chi) = \frac{1}{8}\left({W_\phi}^2 + {W_\chi}^2\right) - \frac{1}{3}W^2. \tag{6}$$

Next we consider the general case with $\Lambda \neq 0$ and we obtain

$$\begin{aligned}
A'' + \Lambda\mathrm{e}^{2A} &= -\frac{2}{3}(\phi'^2 + \chi'^2)\,,\\
A'^2 - \Lambda\mathrm{e}^{2A} &= \frac{1}{6}(\phi'^2 + \chi'^2) - \frac{1}{3}V(\phi, \chi)\,.
\end{aligned} \tag{7}$$

The cosmological constant leads us to define the function which corresponds to the scalar fields ϕ and χ. It means that this function is completely coupled and generally responsible for the cosmological constant. Thus we gain

$$\begin{aligned}
A' &= -\frac{1}{3}W - \frac{1}{3}\Lambda\alpha Z\,,\\
\phi' &= \frac{1}{2}W_\phi + \frac{1}{2}\Lambda\beta Z_\phi\,,\\
\chi' &= \frac{1}{2}W_\chi + \frac{1}{2}\Lambda\beta Z_\chi\,,
\end{aligned} \tag{8}$$

where $Z = Z(\phi, \chi)$ is a new and arbitrary function of the scalar fields. We choose $\alpha = 1$, $\beta = 1 - s$ which are real numbers. The potential $V(\phi, \chi)$ is obtained by (7) and (8):

$$\begin{aligned}
V(\phi, \chi) = {} & \frac{1}{8}(W_\phi + \Lambda(1-s)Z_\phi)(W_\phi + \Lambda(1+3s)Z_\phi)\\
& - \frac{1}{3}(W + \Lambda Z)^2\\
& + \frac{1}{8}(W_\chi + \Lambda(1-s)Z_\chi)(W_\chi + \Lambda(1+3s)Z_\chi)\,.
\end{aligned} \tag{9}$$

By inserting this potential in the equations of motion, one can obtain the following constraint:

$$\begin{aligned}
& W_{\phi\phi}Z_\phi + W_\phi Z_{\phi\phi} + 2\Lambda(1-s)Z_\phi Z_{\phi\phi}\\
& \qquad - \frac{4}{3}WZ_\phi - \frac{4}{3}\Lambda Z Z_\phi = 0\,,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
& W_{\chi\chi}Z_\chi + W_\chi Z_{\chi\chi} + 2\Lambda(1-s)Z_\chi Z_{\chi\chi}\\
& \qquad - \frac{4}{3}WZ_\chi - \frac{4}{3}\Lambda Z Z_\chi = 0\,.
\end{aligned} \tag{11}$$

For simplicity, we consider $Z(\phi, \chi) = W(\phi, \chi)$, because it is difficult to solve these constraints in the general case:

$$\begin{aligned}
& \frac{3}{2}dW_{\phi\phi} - W = 0\,,\\
& \frac{3}{2}dW_{\chi\chi} - W = 0\,,
\end{aligned} \tag{12}$$

where $d = \frac{(1+(1+s)\Lambda)}{(1+\Lambda)}$. These constraints guide us to consider a superpotential with the following form:

$$W = 3a\sinh(b\phi) + 3a\sinh(b\chi)\,, \tag{13}$$

where $b = \pm\sqrt{\frac{2}{3d}}$, and d is positive.

With the solution of (12) in (9) and (8) we get the following potential, fields and $A(y)$, respectively:

$$\begin{aligned}
V = {} & \frac{3}{4}a^2(1+\Lambda)[1 + (1+3s)\Lambda](\cos^2(b\phi) + \cos^2(b\chi))\\
& - 3a^2(1+\Lambda)^2[\sin^2(b\phi) + \sin^2(b\chi) + 2\sin(b\phi)\sin(b\chi)]\,,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\phi(y) &= \frac{1}{b}\sinh^{-1}[\tan(a(1+\Lambda)y)]\,,\\
\chi(y) &= \frac{1}{b}\sinh^{-1}[\tan(a(1+\Lambda)y)]\,,\\
A(y) &= -\ln[sa^2(1+\Lambda)\sec^2(a(1+\Lambda)y)]\,.
\end{aligned} \tag{15}$$

The shape invariance method helps us to investigate the stability condition. By choosing the gauge, we can study the stability of the solution. Due to this solution, first we consider fluctuations of the metric and scalar fields. The perturbed metric is

$$\mathrm{d}s^2 = \mathrm{e}^{2A(y)}(g_{\mu\nu} + \epsilon h_{\mu\nu})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \mathrm{d}y^2\,; \tag{16}$$

we use the coordinate z which is defined by $\mathrm{d}z = \mathrm{e}^{-A(y)}\,\mathrm{d}y$. The corresponding Schrödinger equation is

$$-\frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2} + V(z)\psi(z) = k^2\psi(z)\,, \tag{17}$$

where

$$V(z) = -\frac{9}{4}\Lambda + \frac{9}{4}A'^2(z) + \frac{3}{2}A''(z)\,. \tag{18}$$

In order to discuss the shape invariance method we have to find some first-order equations. For $\Lambda = 0$, the Schrödinger equation factorizes in the form

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + \frac{3}{2}A'(z)\right]\left[-\frac{\mathrm{d}}{\mathrm{d}z} + \frac{3}{2}A'(z)\right]\psi(z) = k^2\psi(z)\,, \tag{19}$$

and the zero mode is the zero-energy state $\psi_0 = \mathrm{e}^{\frac{3}{2}A(z)}$, which identifies the ground state of the quantum mechanical systems.

In general, we discuss the case of $\Lambda \neq 0$, which means that the geometry is dS or AdS. In that case the stability is more involved and the shape invariance version will be interesting. So, we consider $A(z)$ for the case of dS geometry which is obtained by (14),

$$A(z) = -\ln[sa^2(1+\Lambda)\cosh^2(qz)]\,. \tag{20}$$

where $q^2 = \frac{1+\Lambda}{s}$. In the next section we take (17) and discuss the shape invariance method. The shape invariance approach gives information on the energy spectrum and also on the transition between the two geometries from the stability point of view.

3 Shape invariance foundations

If the energy of the ground state is zero, we can factorize the Hamiltonian as [11]

$$H_1(g) = B^\dagger(g)B(g)\,, \tag{21}$$

where g is (are) the real parameter(s) which give(s) us the potential, and $B(g)$ is a first-order differential operator. The ground state of H_1 is annihilated by $B(g)$; the partner Hamiltonian of H_1 will be obtained with reversing the order of B and $B^\dagger$,

$$H_2(g) = B(g)B^\dagger(g)\,, \tag{22}$$

and the spectra of H_1 and H_2 are degenerate. The only difference is that H_1 has a zero-energy state and in general H_2 does not,

$$H_2B = BH_1\,. \tag{23}$$

If we had for $n \geq 0$,

$$H_1{\Psi_n}^{(1)} = E_n^{(1)}{\Psi_n}^{(1)}\,, \tag{24}$$

this would imply that

$$H_2\big(B{\Psi_n}^{(1)}\big) = {E_n}^{(1)}\big(B{\Psi_n}^{(1)}\big)\,. \tag{25}$$

So, the relation between the eigenvalues and eigenfunctions of the two Hamiltonians H_1 and H_2 is

$$\begin{aligned} &E_n^{(2)} = E_{n+1}^{(1)}\,, \quad E_0^{(1)} = 0\,,\\ &\Psi_n^{(2)} \propto A\Psi_{n+1}^{(1)}\,, \end{aligned} \tag{26}$$

where the ground state wavefunction for H_1 (or H_2) can be obtained as follows:

$$B\Psi_0^{(1)}(x) = 0 \Rightarrow \Psi_0^{(1)}(x) = N\exp\left(-\int^x W(y)d(y)\right)\,, \tag{27}$$

$$B^\dagger\Psi_0^{(2)}(x) = 0 \Rightarrow \Psi_0^{(2)}(x) = N\exp\left(+\int^x W(y)d(y)\right)\,. \tag{28}$$

Supersymmetry provides a natural context for understanding the relationship between the states of H_1 and those of H_2 [10], where H_1 and H_2 are partners. If one combines these two operators to

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}\,, \tag{29}$$

then this matrix Hamiltonian can be obtained from the anticommutator $H = \{Q, Q\dagger\}$, where Q and $Q\dagger$ are supercharges, given by

$$Q = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\,, \quad Q^\dagger = \begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix}\,; \tag{30}$$

both Q and $Q^\dagger$ commute with H. In this algebra we have

$$\begin{aligned} &[H, Q] = [H, Q^\dagger] = 0\,,\\ &\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0\,. \end{aligned} \tag{31}$$

The shape invariance is a property that arises when there is an additional relationship between the partner Hamiltonian H_1 and H_2. Suppose that these Hamiltonian are linked by the condition

$$B(g_1)B^\dagger(g_1) = B^\dagger(g_2)B(g_2) + c(g_2)\,, \tag{32}$$

where the real parameters g_1 and g_2 are related by a mapping $f\colon g_1 \longrightarrow g_2$, and $c(g)$ is a c-number that depends on the parameter(s) of the Hamiltonian, when this condition holds the Hamiltonian H_1 is said to be shape invariant. So, in general we can write

$$H_k = B^\dagger(g_k)B(g_k) + c(g_k) + \ldots + c(g_2)\,, \tag{33}$$

where

$$\begin{aligned} &g_{j+1} = f(g_j)\,,\\ &B^\dagger(g_k)H_{k+1} = H_kB^\dagger(g_k)\,. \end{aligned} \tag{34}$$

The ground state of each of these sectors satisfies a first-order equation, namely

$$B(g_k)\Psi_1(x;g_k)=0\,.$$

Now, we study supersymmetry with a central charge. Supersymmetric quantum mechanics [11] can be formulated as a one-dimensional supersymmetric quantum field theory. A bosonic field is, then, a real-valued function of time, and a fermionic field is a Grassman-valued function of time. The $d=1$, $N=1$ superalgebra with a central charge is specified by the following relations:

$$\begin{aligned}\{Q,Q^\dagger\}&=H\,,\\ [H,Q]&=[H,Q^\dagger]=0\,,\\ \{Q,Q\}&=\{Q^\dagger,Q^\dagger\}=C\,,\end{aligned}\tag{35}$$

where Q and C are the supercharge and central charge, respectively; (35) implies $[Q,C]=[Q^\dagger,C]=0$. To realize the algebra (35), we represent the supercharges as matrices:

$$Q=\begin{pmatrix}\lambda & 0\\ B & -\lambda\end{pmatrix},\quad Q^\dagger=\begin{pmatrix}\lambda & B^\dagger\\ 0 & -\lambda\end{pmatrix},\tag{36}$$

where λ is the real part of a c-number. This approach is meant first to present an implementation of this algebra in a two-sector model, and then to generalize this construction to a four-sector model.

Then the corresponding Hamiltonian and central charge are determined by the superalgebra to be, respectively,

$$\begin{aligned}H&=\begin{pmatrix}B^\dagger B+2\lambda^2 & 0\\ 0 & BB^\dagger+2\lambda^2\end{pmatrix},\\ C&=\begin{pmatrix}2\lambda^2 & 0\\ 0 & 2\lambda^2\end{pmatrix},\quad C\geq 0\,;\end{aligned}\tag{37}$$

note that the central charge has only non-negative values in this construction [7, 8].

To construct a model with four sectors, one can concentrate on a two-sector model. It has supercharges

$$\begin{aligned}Q&=\begin{pmatrix}-\lambda_1 & 0 & 0 & 0\\ B_1 & \lambda_1 & 0 & 0\\ 0 & 0 & -\lambda_3 & 0\\ 0 & 0 & B_3 & \lambda_3\end{pmatrix},\\ Q^\dagger&=\begin{pmatrix}-\lambda_1 & {B_1}^\dagger & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & -\lambda_3 & {B_3}^\dagger\\ 0 & 0 & 0 & \lambda_3\end{pmatrix}.\end{aligned}\tag{38}$$

By using (35) and (38) for the four sectors we obtain

$$H=\begin{pmatrix}{B_1}^\dagger B_1+2{\lambda_1}^2 & 0 & 0 & 0\\ 0 & B_1{B_1}^\dagger+2{\lambda_1}^2 & 0 & 0\\ 0 & 0 & {B_3}^\dagger B_3+2{\lambda_3}^2 & 0\\ 0 & 0 & 0 & B_3{B_3}^\dagger+2{\lambda_3}^2\end{pmatrix},\tag{39}$$

and

$$C=\begin{pmatrix}2{\lambda_1}^2 & 0 & 0 & 0\\ 0 & 2{\lambda_1}^2 & 0 & 0\\ 0 & 0 & 2{\lambda_3}^2 & 0\\ 0 & 0 & 0 & 2{\lambda_3}^2\end{pmatrix}.\tag{40}$$

As we see the sectors one and two are degenerate, with energies bounded from below by $2\lambda_1^2$, and sectors three and four are degenerate, with energies bounded from below by $2\lambda_3^2$. The only exceptions are that sectors one and three each have states that saturate their respective energy bounds while the even sectors do not, and this suggest an enhanced algebraic structure. Therefore, we have to define the shift operator S for the four sectors in order to relate sector two to sector three with the following form:

$$S\equiv\begin{pmatrix}0 & 0 & 0 & 0\\ B_1 & 0 & 0 & 0\\ 0 & D & 0 & 0\\ 0 & 0 & B_3 & 0\end{pmatrix}.\tag{41}$$

Also, this operator relates even sectors in the Hamiltonian to BPS states. We can choose D such that the shape invariance condition and $[H,S]=0$ are satisfied. For this, we suppose that there is a unitary transformation which is represented by an operator Ω such that $B_3=\Omega^\dagger B_1\Omega$. And also, we use a unitary operator U such that $U^2=\Omega$, and the conserved shift operator takes the form

$$S\equiv\begin{pmatrix}0 & 0 & 0 & 0\\ B_1 & 0 & 0 & 0\\ 0 & U^\dagger B_1 U & 0 & 0\\ 0 & 0 & {U^\dagger}^2 B_1 U^2 & 0\end{pmatrix}.\tag{42}$$

From conservation of S, one can obtain the shape invariance relation,

$$\begin{aligned}B_1{B_1}^\dagger-U^\dagger{B_1}^\dagger B_1U&=k\,,\\ 2{\lambda_3}^2&=2{\lambda_1}^2+k+U^\dagger kU\,,\end{aligned}\tag{43}$$

where k is a c-number and U implements a shift in the parameter of the theory; in particular we have

$$H=S^\dagger S+F\,,\tag{44}$$

where F is a diagonal matrix that one can determine:

$$F=\begin{pmatrix}2{\lambda_1}^2 & 0 & 0 & 0\\ 0 & 2{\lambda_1}^2+k & 0 & 0\\ 0 & 0 & 2{\lambda_1}^2+k+U^\dagger kU & 0\\ 0 & 0 & 0 & H_4\end{pmatrix}.\tag{45}$$

In the first three sectors, the energies are constrained by a Bogmol'nyi bound, $H_k\geq(F)_{kk}$, because each of the first sector has to be degenerate with the Bogmol'nyi-saturating ground state of one of the first three sectors. The constants in F represent not only the Bogmol'nyi bounds of the various sectors, but also the first three energy eigenvalues of the original Hamiltonian.

In the next section, we shall apply the above information to a bent brane with two scalar fields.

4 Calculation of energy spectrum with the shape invariance method

The algebra we have described gives a natural framework for understanding the origins of shape invariance. Also the study of shape invariance solutions can be done by the factorization method. Our aim is to solve and discuss the stability of a bent brane in different geometries and with a non-zero cosmological constant. In the previous section we have done the perturbation to the metric and fields and achieved the corresponding Schrödinger equation which was the second-order equation. Here we factorize this equation to the first-order equations which are raising and lowering operators and generate the algebra. From these first-order equations we easily discussed the energy spectrum and also the stability of the system in the transition to different geometries. So, first we have to factorize the bent brane Hamiltonian.

By using (18) and substituting it into (17) we have

$$-\frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2}+\left(\frac{9}{4}(4q^2-\Lambda)-12q^2\sec h^2(qz)\right)\psi(z)=k^2\psi(z). \tag{46}$$

For simplicity we take $z=\frac{x}{q}$, and we have

$$H=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-g(g+1)\sec h^2(x)+g^2\,, \tag{47}$$

where $g=3$.

Now we are going to factorize H in terms of the lowering and raising operators, respectively,

$$\begin{aligned}B&=-\frac{\mathrm{d}}{\mathrm{d}x}-g\tanh(x)\,,\\ B^\dagger&=\frac{\mathrm{d}}{\mathrm{d}x}-g\tanh(x)\,,\end{aligned} \tag{48}$$

and one can obtain the paired Hamiltonians

$$\begin{aligned}H_1=B^\dagger B&=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-g(g+1)\sec h^2(x)+g^2\,,\\ H_2=BB^\dagger&=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-g(g-1)\sec h^2(x)+g^2\,,\end{aligned} \tag{49}$$

where

$$H_2(g)=H_1(g-1)+(2g-1)\,. \tag{50}$$

This relation shows us there is a shape invariance condition with $c(g)=2g-1$.

In the case of a central charge, we choose the unitary operator U as follows:

$$U=\exp\left(\frac{\partial}{\partial g}\right),\quad U^\dagger=\exp\left(-\frac{\partial}{\partial g}\right), \tag{51}$$

where

$$U^\dagger f(g)U\longrightarrow f(g-1)\,.$$

From (42)–(44) we have

$$S^\dagger S=\begin{pmatrix}{B_1}^\dagger B_1 & 0 & 0 & 0\\ 0 & U^\dagger {B_1}^\dagger B_1 U & 0 & 0\\ 0 & 0 & \Omega^\dagger {B_1}^\dagger B_1\Omega & 0\\ 0 & 0 & 0 & H_4\end{pmatrix}, \tag{52}$$

with

$$\begin{aligned}{B_1}^\dagger B_1&=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-g(g+1)\sec h^2(x)+g^2,\\ U^\dagger {B_1}^\dagger B_1 U&=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-(g-1)(g)\sec h^2(x)+(g-1)^2,\\ \Omega^\dagger {B_1}^\dagger B_1\Omega&=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-(g-2)(g-1)\sec h^2(x)+(g-2)^2.\end{aligned} \tag{53}$$

Also, by using (44) and (45), one can obtain F as follows:

$$F=\begin{pmatrix}\frac{-9\Lambda}{4q^2} & 0 & 0 & 0\\ 0 & \frac{-9\Lambda}{4q^2}+2g-1 & 0 & 0\\ 0 & 0 & \frac{-9\Lambda}{4q^2}+4g-4 & 0\\ 0 & 0 & 0 & H_4\end{pmatrix}. \tag{54}$$

Therefore, the energy spectrum of H_1 is

$$\begin{aligned}E_0^{(1)}&=-\frac{9\Lambda}{4}\,,\\ E_1^{(1)}&=-\frac{9\Lambda}{4}+5q^2,\\ E_2^{(1)}&=-\frac{9\Lambda}{4}+8q^2,\\ E_3^{(1)}&=H_4\,.\end{aligned} \tag{55}$$

Also from (27) we obtain

$$\begin{aligned}\psi_0^{(1)}(z)&\propto\sec h^3(qz)\,,\\ \psi_1^{(1)}(z)&\propto\sec h^{-3}(qz)\,,\end{aligned} \tag{56}$$

where they are the two first eigenstates of the original Hamiltonian.

We note that for the AdS$_4$ case the cosmological constant is negative, so all states have positive eigenvalues. In this case the solutions are stable. In case of the dS$_4$ geometry the cosmological constant is positive, so some states have a negative eigenvalue. It means that we have some tachyonic state with negative energy. Also, in case of M_4 the cosmological constant is zero, so all states have a positive mode (except one of them that is the zero mode). These results show that the transition from AdS$_4$ and M$_4$ to dS$_4$ geometry is not stable.

5 Conclusion

In this paper we have considered models described by coupled real scalar fields in five-dimensional space-time with dS geometry. This model leads us to introduce two couple functions $W(\phi,\chi)$ and $Z(\phi,\chi)$. We related the functions W and Z to the warp factor. By entering fluctuations of metric and fields, we obtained a Schrödinger equation. For solving this second-order differential equation, we dealt mainly with the possibility of obtaining first-order equations in a braneworld scenario driven by scalar fields ϕ and χ, with embedded geometry of the AdS, M, or dS type. The factorized Hamiltonian for the bent brane leads us to investigate the shape invariance method with considering the central extended algebra. Finally we used the shape invariance method and obtained the energy spectrum for all states. Also, we have shown that the transition from the AdS_4 and M_4 to the dS_4 geometry is not stable. For simplicity we considered $W(\phi,\chi) = Z(\phi,\chi)$ which is defined by $W_{\phi\chi} = 0$. It may be interesting to solve the case of $W_{\phi\chi} \neq 0$.

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